

Solving beam deflection and rotation using higher-order moments of force

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ABSTRACT

Traditional methods for determining beam deflection, such as the double-integration and moment-area method, involve multiple steps including indefinite integration, diagram construction, and evaluation of constants. This paper introduces a novel analytical framework based on higher-order moments of force to streamline the analysis of beam rotation and deflection. In some fields, higher-order moments, such as area moment of inertia and mass moment of inertia, are well-established, yet no formal definition currently exists in structural engineering for higher-order moments of force, particularly in the context of elastic beam analysis. This study defined and applied second- and third-order moments of force to derive closed-form expressions for slope and deflection without requiring auxiliary diagrams or segment-by-segment integration. The method was grounded in Euler–Bernoulli beam theory and was applicable to a wide range of loading types, including concentrated loads, distributed loads, and concentrated moments. Though the proposed higher-order moments lacked physical interpretation, their mathematical validity was demonstrated through rigorous derivation and consistency with classical solutions. This work offered a generalizable, efficient alternative to traditional methods and represented an initial effort to formalize the use of higher-order moments of force in structural engineering analysis.

INTRODUCTION

Traditional analytical methods for determining beam deflections—such as the double-integration method, the moment-area theorem, conjugate-beam theory, and the method of superposition—are well documented in classical structural texts (Wang & Eckel, 1957; Timoshenko, 1953; Hibbeler, 2020). While each of these approaches reliably predicts deflections and rotations, they often demand multiple indefinite integrations, auxiliary diagrams (e.g. bending-moment diagram and conjugate-beam sketches), and segment-by-segment bending-moment expressions. These steps can be cumbersome, error-prone, and time-consuming, especially

for beams subjected to complex or discontinuous loading (Jong, 2009).

In search of more direct solutions, Grandin and Rencis (2006) introduced a “Method of Segments” that applies axial and torsion bar force–deformation formulas directly to beam problems. By dividing a beam into segments of uniform properties, their formulation yields end-slope and end-displacement expressions without further integration or constant-of-integration resolution. Jong (2009) further refined this technique by limiting segmentation only to points of slope discontinuity, enabling broader applicability across varied beam configurations. While these direct-equation methods reduce procedural complexity, they remain limited in several ways: (1) they require manual identification of segment boundaries, (2) they depend on the load type, and (3) they do not generalize well to arbitrary or piecewise-continuous loading profiles.

To address these challenges, this paper introduces a more flexible and broadly applicable framework based on higher-order moments of force. Unlike existing approaches, the proposed method requires no segmentation except at changes in flexural rigidity or internal releases, and eliminates the need for auxiliary diagrams and case-specific adjustments by relying on generalized expressions for slope and deflection.

However, no formal definitions or physical interpretations currently exist in structural engineering for second- or third-order moments of force. This absence reveals a theoretical gap and presents an opportunity for methodological innovation. Although this work does not seek to establish a physical interpretation of these higher-order moments, it instead defines them mathematically to support the proposed method.

The framework defines second- and third-order moments of internal and external forces about a reference point and demonstrates how these can be used to derive closed-form expressions for angular rotation and vertical deflection. These

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Date received: 02 August 2025

Dates revised: 25 January 2026; 29 March 2026; 16 April 2026; 25 July 2026

Date accepted: 27 September 2026

DOI: <https://doi.org/10.54645/2026192LPE-21>

KEYWORDS

beam deflection; elastic curve, Euler–Bernoulli beam theory, higher-order moments of force, slope and deflection analysis, structural analysis, analytical method

expressions apply to a wide range of loading conditions—including point loads, distributed loads, and concentrated moments—regardless of whether the beam geometry or loading is continuous or discontinuous.

This work is grounded in the Euler–Bernoulli beam theory, assuming linear elasticity, small deformations, and piecewise-constant flexural rigidity. Under these conditions, the proposed formulation offers a mathematically rigorous, physically consistent, and computationally efficient alternative to traditional deflection analysis. The following sections define the necessary mathematical tools, develop the core theorems, and demonstrate their application through worked examples.

MATERIALS AND METHODS

This study was conducted as a theoretical and analytical investigation within the framework of Euler–Bernoulli beam theory, which assumes linear elasticity, small deformations, and piecewise-constant flexural rigidity.

The proposed method introduces mathematically defined second- and third-order moments of force, which are then used to derive two core theorems: Theorem I (Rotation Theorem) and Theorem II (Deflection Theorem).

Defining Higher-Order Moments of Force

In its most basic form, a moment is a mathematical expression representing the product of a physical quantity, Q (such as concentrated force or electrical charge), and its distance, r , raised to a power n :

$$\mu^{(n)} = r^n Q \quad (1)$$

If the quantity Q is not concentrated but distributed along a region of the beam, a continuous representation is used. Let $\rho(r)$ be the quantity as a function of distance r . The higher-order moment may be interpreted as the accumulation of differential contributions of the form $d\mu^{(n)} = r^n \rho(r) dr$ over the region of interest. Accordingly, the n th-order moment of the distributed force over the interval is defined by:

$$\mu^{(n)} = \int r^n \rho(r) dr \quad (2)$$

While the specific form of these equations may vary depending on the complexity of the relationship between the quantity and distance, they will always include the fundamental term $r^n Q$ or its equivalent. The exponent n defines the order of the moment, which generally influences its physical interpretation. Physical interpretations of second-order and higher-order moments are well established in other fields, such as mechanics and dynamics. In mechanics, for example, the second moment of area (commonly referred to as the area moment of inertia or simply moment of inertia), describes how a cross-sectional area is distributed relative to a given axis and quantifies a section's resistance to bending. It is given by:

$$I = \int d^2 dA \quad (3)$$

where I is the second-order moment of area, and d is the perpendicular distance from the area element dA to the reference axis.

A different type of second-order moment also appears in rotational dynamics: the mass moment of inertia, which describes how mass is distributed relative to an axis of rotation and quantifies an object's resistance to changes in angular motion. It is given by:

$$I = \int r^2 dm \quad (4)$$

where I is the mass moment of inertia, r is the perpendicular distance from the mass element, dm , to the axis of rotation.

In structural mechanics, where a concentrated force, F , located at a perpendicular distance, d , from a point of interest as shown in Fig. 1, the first-order moment (or simply, moment), M , about that point is given by:

$$M = Fd \quad (5)$$

The first-order moment of force represents the rotational effect of a force on a point or body. In contrast, higher-order moments of force lack a defined physical interpretation in structural mechanics. Nevertheless, extending the logic that is well-established in other fields, and for the purpose of this paper, higher-order moments of concentrated force, F , are mathematically defined as:

$$M^{(n)} = Fd^n \quad (6)$$

If the force is not concentrated but distributed along a region of the beam, a continuous representation is used. Let $w(x)$ be the load intensity as a function of distance x . The higher-order moment may be interpreted as the accumulation of differential contributions of the form $[dM^{(n)}] = [w(x) dx] \cdot [d^n]$, which can be rewritten as $dM^{(n)} = w(x) d^n dx$. Then, the n th-order moment of the distributed force over the interval $[a, b]$ is defined by:

$$M^{(n)} = \int_a^b w(x) (d)^n dx \quad (7)$$

Where:

- n is the order of the moment,
- $w(x)$ is the distributed load as a function of position x ,
- x is the distance of a strip of distributed load, measured from the origin (see Fig. 2),
- d is the moment arm of a strip of the distributed load, equal to $(x_L - x)$ (see Fig. 2),
- $[a, b]$ is the span of the load distribution.

This definition forms the basis for evaluating second- and third-order moments of force, which are central to the method proposed in this paper.

Concentrated Forces

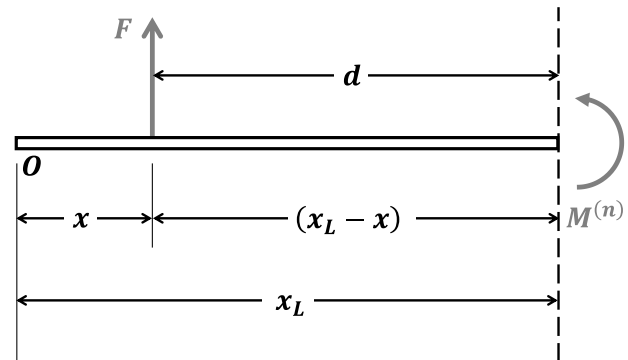


Figure 1: Concentrated Load

For a concentrated load F shown in Fig. 1, located at a distance x , from a point of origin, O , the higher-order moments about a point of interest located at a distance x_L are directly computed as:

$$M^{(1)} = F (x_L - x) \quad (8)$$

$$M^{(2)} = F (x_L - x)^2 \quad (9)$$

$$M^{(3)} = F (x_L - x)^3 \quad (10)$$

Uniformly Distributed Loads

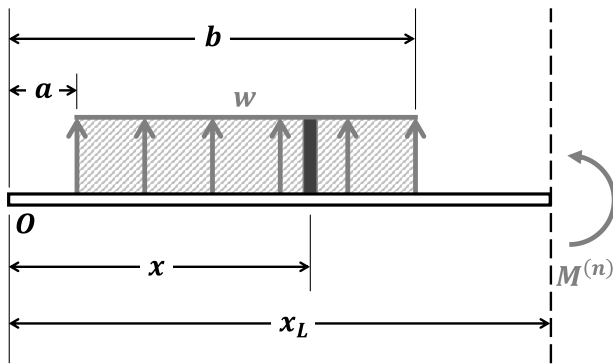


Figure 2: Uniformly Distributed Load

Consider a uniformly distributed load shown in Fig. 2, with intensity w extending from a to b , measured from a reference point, O . The load function is:

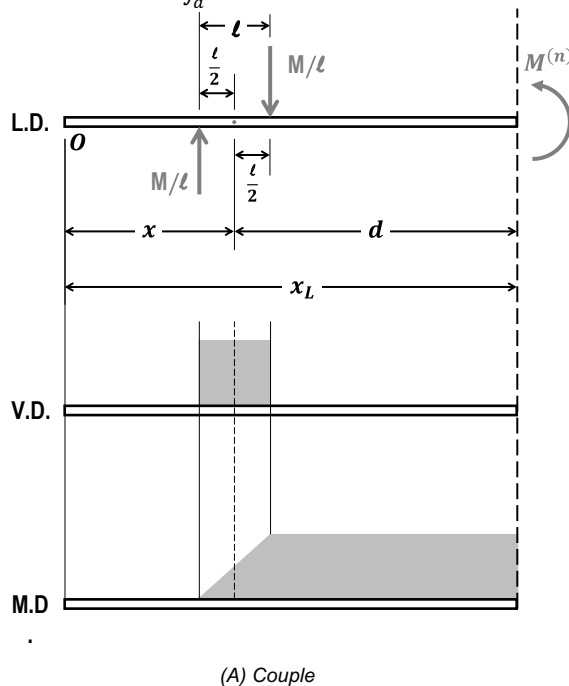
$$w(x) = w \rightarrow \text{constant} \quad (11)$$

Substituting into Eq. (7), the first three orders of moments about the point of interest, located at x_L from the reference point O , are:

$$M^{(1)} = \int_a^b w (x_L - x) dx \quad (12)$$

$$M^{(2)} = \int_a^b w (x_L - x)^2 dx \quad (13)$$

$$M^{(3)} = \int_a^b w (x_L - x)^3 dx \quad (14)$$



Linearly Varying Loads

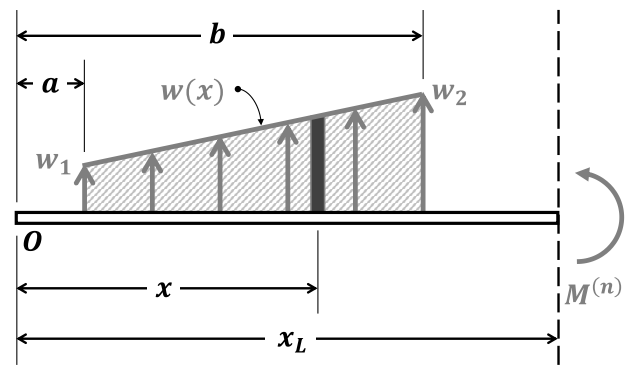


Figure 3: Linearly Varying Load

For a load that varies linearly from w_1 at $x = a$ to w_2 at $x = b$, as shown in Fig. 3, the intensity function is:

$$w(x) = w_1 + \left(\frac{w_2 - w_1}{b - a} \right) (x - a) \quad (15)$$

This linear function is then substituted into Eq. (7) to calculate the higher-order moments:

$$M^{(n)} = \int_a^b \left[w_1 + \left(\frac{w_2 - w_1}{b - a} \right) (x - a) \right] (x_L - x)^n dx \quad (16)$$

Worked examples for second- and third-order moments under such loading are presented in the application section.

Concentrated Moments and Equivalence

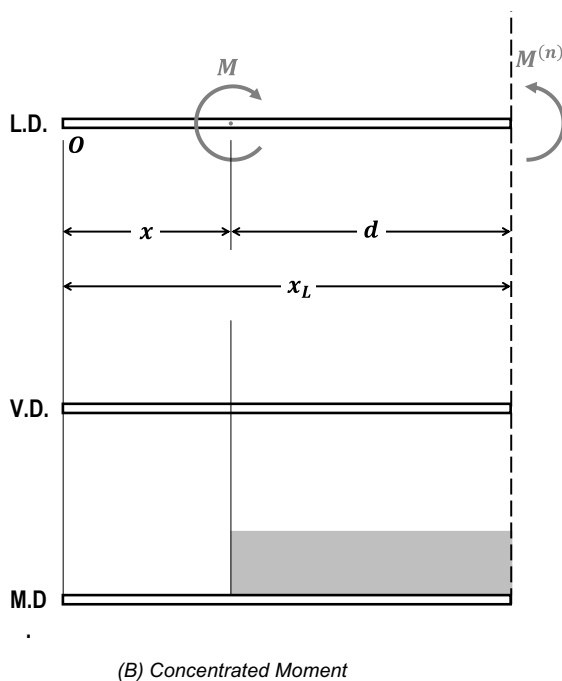


Figure 4: Shear and Moment Diagrams

Fig. 4(A) and Fig. 4(B) illustrate that a beam subjected to a couple and a beam subjected to a concentrated moment do not produce identical shear and bending moment diagrams. While both cases resulted in zero resultant force, the shear diagram of the beam with a couple in Fig. 4(A) showed that a segment of the beam contained internal shear forces, whereas the beam with a concentrated moment, as shown in Fig. 4(B), had no shear forces along its entire length. Additionally, the bending moment of the beam with a couple in Fig. 4(A) exhibited a gradual change along the segment between the applied forces. In contrast, for the beam in Fig. 4(B), the moment diagram experienced a sudden change, or "jump", at

the point of application of the concentrated moment and remained constant throughout the rest of the beam's length.

The differences between the two systems arose from the nature of the couple, which required a finite distance between two parallel forces, whereas concentrated moment was applied at a single point. As the distance between the couple decreased and approached zero, the behavior of the two systems became increasingly similar (Pytel & Kiusalaas, 2014; Meriam & Kraige, 2012). In Fig. 4(A), the gradual change in the bending moment for the beam subjected to a couple could be eliminated, allowing it to replicate the "jump"

behavior of the concentrated moment in Fig. 4(B). This was achieved by taking the limit of the bending moment, M , at the point of interest as the distance, l , approached zero. To apply this concept to the couple in Fig. 4(A), the magnitude of each force was first determined such that the couple produced a moment equal in magnitude to M in Fig. 4(B), given the distance l between the two equal and opposite forces. For this purpose, the magnitude of the forces are expressed as a function of M and the distance l :

$$F = \frac{M}{l} \quad (17)$$

By placing their midpoint at a distance x from the origin, then the expression of the moment of order n due to this couple about the point of interest x_L is written as:

$$M^{(n)} = \frac{M}{l} \left(x_L - x + \frac{l}{2} \right)^n - \frac{M}{l} \left(x_L - x - \frac{l}{2} \right)^n \quad (18)$$

To achieve the same effect as that of the concentrated moment, as explained, the limit of Eq. (18) is evaluated as l approached zero (0) for $n = \{1, 2, 3\}$:

$$M^{(1)} = M \quad (19)$$

$$M^{(2)} = 2M(x_L - x) \quad (20)$$

$$M^{(3)} = 3M(x_L - x)^2 \quad (21)$$

As shown in Eq. (19), first-order moments of concentrated moment load, which act as rotational forces rather than linear forces, are not affected by their location. Intuitively, the first-order moment of a concentrated moment load is equal to its magnitude, as there is no moment arm involved. In contrast, when evaluating higher-order moments of a concentrated moment load, the resulting values become dependent on the load's position.

The generalized concept of higher-order moments of force provides a mathematical foundation for deriving expressions for beam rotation and deflection. While the first moment of force corresponds to its rotational effect on a point, the second- and third-order moments—though lacking physical interpretation—prove analytically valuable. These formulations will be used in the next section to establish Theorems I and II.

Development of Proof of Theorems I and II

Proof of Theorem I

Theorem I (Rotation Theorem).

The change in angle of rotation between two arbitrary points A and B on a continuous elastic curve is equal to one-half the summation of the second moments of applied and reactive forces between A and B, taken about point B, divided by the beam's flexural rigidity EI :

$$\theta_B - \theta_A = \frac{1}{2EI} \sum M_B^{(2)} \quad (22)$$

Consider beam segment AB as a Euler-Bernoulli beam of length L , with constant flexural rigidity EI , and arbitrary loading distributions (see Fig. 5). Let the beam carry:

- a constant distributed load $w_1(x)$,
- a linearly varying load $w_2(x)$,
- an arbitrary load $w_n(x)$,
- as well as internal forces (shear and moment) at points A and B, which represent concentrated force-type and moment-type loads.

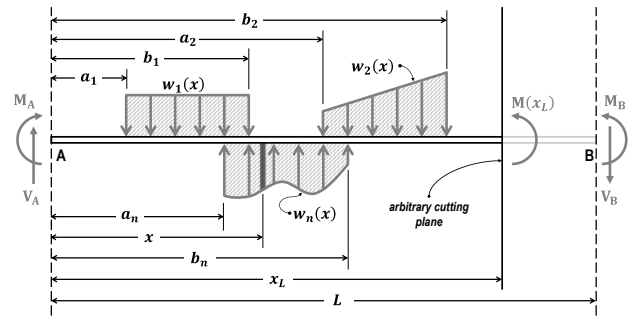


Figure 5: Beam section under a set of different types of loads

Where:

- L is the total span of beam segment AB
- x_L is the position of an arbitrary cutting plane measured from the point of origin A, where $0 \leq x_L \leq L$;
- x is the arbitrary distance of the load from the point of origin A, where $0 \leq x \leq x_L$;
- $M(x_L)$ is the moment function, expressed as a function of x_L , taken at an arbitrary point of interest.

From beam theory, the relationship between bending moment and curvature is:

$$\frac{d^2y}{(dx_L)^2} = \frac{M(x_L)}{EI} \quad (23)$$

Integrating both sides once with respect to x_L gives the angle of rotation:

$$\theta(x_L) = \int \frac{M(x_L)}{EI} dx_L + C_1 \quad (24)$$

To construct a general expression for the bending moment over the entire beam segment AB , a movable cutting plane is introduced at a distance x_L from the origin A. The cutting plane is allowed to vary continuously along the beam segment, enabling the derivation of a moment function $M(x_L)$ that is valid for all $0 \leq x_L \leq L$.

Macaulay's method is employed to formalize this approach and to express load contributions within a single unified equation (Beer et al., 2012; Timoshenko & Gere, 1991). Under this formulation, only loads located to the left of the cutting plane contribute to the internal moment. This condition is automatically enforced through the Macaulay bracket $\langle x_L - x \rangle$, which evaluates to zero when the load position exceeds the cutting plane location.

Accordingly, two independent spatial variables are introduced: x_L , which defines the position of the cutting plane, and x , which defines the position of the applied loads. This distinction is essential for expressing the moment arm, which is the distance between the cutting plane and each load, given by $\langle x_L - x \rangle$, which governs the contribution of the load to the moment.

The bending moment equation for the beam segment AB expressed as a function of x_L becomes:

$$M(x_L) = V_A x_L + M_A - \int_{a_1}^{b_1} w_1(x) \langle x_L - x \rangle dx - \int_{a_2}^{b_2} w_2(x) \langle x_L - x \rangle dx + \int_{a_n}^{b_n} w_n(x) \langle x_L - x \rangle dx \quad (25)$$

Since the loads are arbitrary, meaning that multiple loads of the same type may exist, it is necessary to follow a systematic approach to derive a general expression for the moment contributed by each load category. For illustration, consider V_A a representative concentrated load. In Eq. (25), its moment about the arbitrary cutting plane (see Fig. 5), located at x_L is given by $V_A x_L$. Other concentrated loads may also be present within the beam segment, including externally applied loads, support reactions, and internal shear forces at a cut section. These are treated equivalently, as each contributes to the moment through a force acting at a distance from the point of interest. For any concentrated load, F , located at any arbitrary position x where $0 \leq x \leq x_L$, the moment about an arbitrary point of interest is given by $F \langle x_L - x \rangle$. To account for multiple concentrated loads along the span, the total contribution

of this load type can be expressed using summation notation, $\sum_i F_i \langle x_L - x_i \rangle$.

The same analogy can be applied to concentrated moments. In Eq. (25), M_A is the representative of this load type and its moment about the cutting plane is the moment itself, M_A . Note that the same load type may exist within the beam segment, including externally applied moments, moments from support reactions, and internal bending moment at cut sections. To account for multiple concentrated moment loads along the span, the total contribution of this load type can be expressed using summation notation, $\sum_j M_j$.

This expression can be extended to include other load types to form the general moment expression about the point of interest (x_L). Using sigma notation, the total moment may be written as:

$$M(x_L) = \sum_i F_i \langle x_L - x_i \rangle + \sum_j M_j + \sum_k \left[\int_{a_k}^{b_k} w_k(x) \langle x_L - x \rangle dx \right] \quad (26)$$

where:

- F_i are concentrated forces at positions x_i , where $0 \leq x_i \leq x_L$,
- M_j are concentrated moments at positions x_j , where $0 \leq x_j \leq x_L$,
- $w_k(x)$ are distributed loads acting over intervals $[a_k, b_k]$,
- $\langle \cdot \rangle$ denotes the Macaulay bracket.

Substituting Eq. (26) into Eq. (24) and integrating, the equation for the angle of rotation becomes:

$$\theta(x_L) = \frac{1}{EI} \left[\sum_i \frac{F_i \langle x_L - x_i \rangle^2}{2} + \sum_j M_j \langle x_L - x_j \rangle + \sum_k \left[\int_{a_k}^{b_k} \frac{w_k(x) \langle x_L - x \rangle^2}{2} dx \right] \right] + C_1 \quad (27)$$

Factoring out $1/2$ from Eq. (27), the equation becomes:

$$\theta(x_L) = \frac{1}{2EI} \left[\sum_i F_i \langle x_L - x_i \rangle^2 + \sum_j 2M_j \langle x_L - x_j \rangle + \sum_k \left[\int_{a_k}^{b_k} w_k(x) \langle x_L - x \rangle^2 dx \right] \right] + C_1 \quad (28)$$

If the arbitrary distance x_L is set to zero, then by the Macaulay function, $\langle x_L - x \rangle = 0$ (Beer et al., 2012; Timoshenko & Gere, 1991). This eliminates the definite integration,

$$\theta(0) = \frac{1}{2EI} [0 + 0 + 0] + C_1 \quad (29)$$

The constant of integration C_1 can be determined:

$$C_1 = \theta(0) \quad (30)$$

Substituting Eq. (30) into Eq. (28),

$$\theta(x_L) = \frac{1}{2EI} \left[\sum_i F_i \langle x_L - x_i \rangle^2 + \sum_j 2M_j \langle x_L - x_j \rangle + \sum_k \left[\int_{a_k}^{b_k} w_k(x) \langle x_L - x \rangle^2 dx \right] \right] + \theta(0) \quad (31)$$

Note that points A and B, are located at $x_L = 0$ and $x_L = L$, respectively. Thus, it can be written:

$$\theta(0) = \theta_A \quad (32)$$

$$\theta(L) = \theta_B \quad (33)$$

Substituting Eq. (32) and Eq. (33) into Eq. (31),

$$\theta_B = \frac{1}{2EI} \left[\sum_i F_i \langle L - x_i \rangle^2 + \sum_j 2M_j \langle L - x_j \rangle + \sum_k \left[\int_{a_k}^{b_k} w_k(x) \langle L - x \rangle^2 dx \right] \right] + \theta_A \quad (34)$$

Furthermore, notice that the terms within the bracket are the second-order moments for each type of load, as previously defined.

Eq. (34) can therefore be rewritten as:

$$\theta_B = \frac{1}{2EI} \sum M_B^{(2)} + \theta_A \quad (35)$$

Finally, rearranging the terms will yield the equation for the first theorem. See Eq. (22).

Proof of Theorem II

Theorem II (Deflection Theorem).

The difference in transverse deflection between two arbitrary points A and B, measured perpendicular to the beam's original axis, on a continuous elastic curve is equal to the product of the rotation at A and the distance between A and B, plus one-sixth the third moment of the applied forces and reactions between A and B, taken about point B, divided by the flexural rigidity EI:

$$\delta_B - \delta_A = \theta_A L + \frac{1}{6EI} \sum M_B^{(3)} \quad (36)$$

Starting from the curvature-moment relationship as given in Eq. (23):

$$\frac{d^2 y}{(dx_L)^2} = \frac{M(x_L)}{EI}$$

Integrating once to obtain the slope $\theta(x_L)$, which was already defined in Eq. (31):

$$\theta(x_L) = \frac{1}{2EI} \left[\sum_i F_i \langle x_L - x_i \rangle^2 + \sum_j 2M_j \langle x_L - x_j \rangle + \sum_k \left[\int_{a_k}^{b_k} w_k(x) \langle x_L - x \rangle^2 dx \right] \right] + \theta(0)$$

Integrating again yields the transverse deflection $y(x_L)$:

$$y(x_L) = \int \theta(x_L) dx_L \quad (37)$$

Substituting Eq. (31) into Eq. (37):

$$y(x_L) = \int \left[\frac{1}{2EI} \left[\sum_i F_i \langle x_L - x_i \rangle^2 + \sum_j 2M_j \langle x_L - x_j \rangle + \sum_k \left[\int_{a_k}^{b_k} w_k(x) \langle x_L - x \rangle^2 dx \right] \right] + \theta(0) \right] dx_L + C_2 \quad (38)$$

Simplifying further:

$$y(x_L) = \frac{1}{2EI} \left[\sum_i \frac{F_i \langle x_L - x_i \rangle^3}{3} + \sum_j M_j \langle x_L - x_j \rangle^2 + \sum_k \left[\int_{a_k}^{b_k} \frac{w_k(x) \langle x_L - x \rangle^3}{3} dx \right] \right] + \theta(0) \langle x_L \rangle + C_2 \quad (39)$$

Factoring out $1/3$ from Eq. (39), the equation becomes:

$$y(x_L) = \frac{1}{6EI} \left[\sum_i F_i \langle x_L - x_i \rangle^3 + \sum_j 3M_j \langle x_L - x_j \rangle^2 + \sum_k \left[\int_{a_k}^{b_k} w_k(x) \langle x_L - x \rangle^3 dx \right] \right] + \theta(0) \langle x_L \rangle + C_2 \quad (40)$$

If the arbitrary distance x_L is set to zero, then by the Macaulay function, $\langle x_L - x \rangle = 0$. This eliminates the definite integration and some other terms,

$$y(0) = \frac{1}{6EI} [0 + 0 + 0] + 0 + C_2 \quad (41)$$

The constant of integration C_2 can be determined:

$$C_2 = y(0) \quad (42)$$

Substituting Eq. (42) into Eq. (40),

$$y(x_L) = \frac{1}{6EI} \left[\sum_i F_i \langle x_L - x_i \rangle^3 + \sum_j 3M_j \langle x_L - x_j \rangle^2 + \sum_k \left[\int_{a_k}^{b_k} w_k(x) \langle x_L - x \rangle^3 dx \right] \right] + \theta(0) \langle x_L \rangle + y(0) \quad (43)$$

Note that in the previous section, $\theta(0)$ and $\theta(L)$ have already been defined as θ_A and θ_B , respectively. Using the same logic used for this, $y(0)$ and $y(L)$ are obtained:

$$y(0) = \delta_A \quad (44)$$

$$y(L) = \delta_B \quad (45)$$

Substituting Eq. (44), Eq. (45), and Eq. (32) into Eq. (43),

$$\delta_B = \frac{1}{6EI} \left[\sum_i F_i \langle L - x_i \rangle^3 + \sum_j 3M_j \langle L - x_j \rangle^2 + \sum_k \left[\int_{a_k}^{b_k} w_k(x) \langle L - x \rangle^3 dx \right] \right] + \theta_A \langle L \rangle + \delta_A \quad (46)$$

Furthermore, notice that the terms within the bracket are the third-order moments for each type of load, as previously defined.

Eq. (46) can therefore be rewritten as:

$$\delta_B = \frac{1}{6EI} \sum M_B^{(3)} + \theta_A L + \delta_A \quad (47)$$

Finally, rearranging the terms will yield the equation for the second theorem. See Eq. (36).

Procedure of Analysis

The analytical expressions derived in Theorems I and II form the foundation of a general method for evaluating beam rotation and deflection based on higher-order moments of force. These

expressions are collectively referred to as the elastic curve equations:

- **Theorem I (Rotation):**

$$\theta_B - \theta_A = \frac{1}{2EI} \sum M_B^{(2)}$$

- **Theorem II (Deflection):**

$$\delta_B - \delta_A = \theta_A L + \frac{1}{6EI} \sum M_B^{(3)}$$

The following steps outline how to apply these equations to a wide range of beam problems:

Beam Uniformity and Segmenting

The applicability of the elastic curve equations depends on the homogeneity and uniformity of the beam's cross-section. Therefore, these properties must be verified prior to further analysis. For beams with discrete changes in flexural rigidity, EI , it is standard practice to divide the beam at each point where a change in EI occurs. At each cut, the internal forces, which include the shear and bending moment, must be considered. The elastic curve equations (Theorem I and Theorem II) are applied to each segment independently, accounting for internal forces (shear and bending moment at cut sections).

Non-prismatic beams with continuously varying EI are beyond the scope of this approach.

Load and Reaction Identification

All applied and reactive forces—whether external (support reactions) or internal (i.e. shear force and bending moment at a cut section)—must be clearly identified. Unlike classical methods, this approach does not require auxiliary diagrams (e.g., shear or moment diagrams), as it does not depend on the distribution of internal forces along the beam. Instead, only the values of the internal shear and moment at the cut section are required.

Accordingly, the analysis relies on the accurate definition of:

- all applied loads (point, distributed, varying);
- support reactions;
- shear and bending moments at cut sections; and
- locations and orientations of each force.

For statically indeterminate beams, support reactions may be initially unknown. In such cases, compatibility conditions derived from Theorems I and II are used to generate additional equations for solving the unknowns.

Selecting the Point of Origin

For the purpose of this formulation, a point of origin is selected as a reference point from which the analysis proceeds. This point is denoted as A in Eq. (22) and Eq. (36) for the beam segment (AB) illustrated in Fig. 5. It must be emphasized, however, that this notation does not imply that point (A) is a fixed physical location on the beam. Rather, the designation of the point of origin depends on the selected segment under consideration.

In general, once a segment of the beam is chosen for analysis, the point of origin is defined as the endpoint of that segment located to the left. This convention ensures a consistent direction of evaluation throughout the formulation. While it is often convenient to select the left-most end of the beam as the point of origin, the origin may be located anywhere along the beam, particularly when the beam is analyzed in segments.

For example, in the case of segment (AB), point (A) is designated as the point of origin and point (B) as the point of interest. However, if segment (CD) is selected instead, then point (C), being the left endpoint, is designated as the point of origin, and point (D) becomes the corresponding point of interest.

If the point of origin is chosen within the span of the beam, the beam is conceptually divided at that location, and only the free body diagram (FBD) of the segment to the right of the point of origin is considered. In such cases, the internal forces at the point of division must be included in the analysis.

Defining the Point of Interest

The point of interest is defined as the endpoint of the selected segment located to the right of the point of origin. This point is denoted as (B) in Eq. (22) and Eq. (36) for the beam segment (AB) shown in Fig. 5. Similar to the point of origin, this notation represents a role assigned within the selected segment and does not correspond to a fixed point on the beam.

For consistency in the formulation, the segment bounded by the point of origin and the point of interest is isolated and analyzed, and all forces and moments acting outside this segment are excluded. In general, for any selected segment, the right endpoint is designated as the point of interest. Thus, for segment (AB), point (B) serves as the point of interest, whereas for segment (CD), point (D) assumes this role. This assignment is based solely on the relative positions of the endpoints within the chosen segment.

In addition, the point of interest serves as the reference point at which all higher-order moments are evaluated in the present formulation. Consequently, loads or reactions acting directly at the point of interest do not contribute to the summation of higher-order moments, as their moment arms with respect to this point are zero.

While the point of interest is commonly the location at which the structural response (e.g., deflection or rotation) is evaluated, it is not necessarily the only location where unknown responses may exist. Depending on the problem configuration, unknown values of deflection or rotation may also be associated with the point of origin (see Example 2). The designation of the point of interest is therefore based on its position within the selected segment and its role in the evaluation of higher-order moments, rather than solely on the presence of unknown quantities.

Determining Initial Boundary Constants

Initial boundary constants refer to the slope and deflection at a specified point—preferably at the point of origin, denoted as point A . The initial slope θ_A and initial deflection δ_A are required to apply the elastic curve equations when solving for the slope and deflection at another point of interest, B . These known values are collectively referred to as the initial boundary constants, and they are typically determined from the beam's support conditions or through auxiliary calculations—for example, by placing the point of origin at a location where the slope and deflection are already known or defined (e.g., at the fixed end of a cantilever beam).

In some cases, such as a simply supported beam, the boundary condition may be known as zero deflection at the supports. For these cases, when both deflections at two points are known, Theorem II can be rearranged to solve for the unknown slope θ_A (see Example 3).

Although the unknown value is typically at the point of interest, B , certain problems may require placing the unknown at the point of origin, A , and assigning the known boundary constants to the point of interest, B . The elastic curve equations remain valid in either configuration. However, by convention, the point of origin is always located at the leftmost end of the beam segment under analysis, while the point of interest is placed at the rightmost end. In either case, all higher-order moments are taken about point B .

Sign Conventions

The following sign conventions are adopted to ensure consistency throughout this paper and to minimize ambiguity in the formulation. While readers may be familiar with alternative conventions, such as those used in the double integration method,

a unified system is necessary for the present work, particularly due to the use of higher-order moments.

To provide an intuitive basis, the adopted conventions are aligned with the standard Cartesian coordinate system, where movement to the right is positive, upward movement is positive, and a line with positive slope (rotation) rises upward to the right. This familiar geometric interpretation is used to guide the definition of sign for forces, rotation, and deflection.

The sign convention of moment and its higher-order forms is dictated by the sign of the force from which it is derived. Since the arbitrary point of interest is always located to the right of the applied loads, the moment arm ($x_L - x$) remains positive within the domain of interest. Thus, a positive force (upward) produces a positive moment. This is consistent with the standard convention in beam theory, where a positive moment corresponds to tension at the bottom fiber and compression at the top. Accordingly, this formulation adopts this sign convention for concentrated moments. For higher-order moments, the sign remains consistent with its base form since the moment arm ($x_L - x$) is always positive.

When the point of origin and point of interest are interchanged, the moment arm becomes negative because the reference point shifts to the left of the applied loads. For moments with odd power, such as first-order moments of force (Eq. (8)), third-order moments of force (Eq. (10)), and second-order moment of concentrated moment (Eq. (20)), this does not pose a problem, since the sign of the moment arm is preserved. However, for moments with even power, such as second-order moment of force (Eq. (9)) and third-order moment of concentrated moment (Eq. (21)), the sign is eliminated. This loss of sign leads to incorrect evaluation of rotation and deflection, particularly in cases where multiple load types are present. This is evident when comparing the higher-order moment expressions for concentrated forces and concentrated moments. For a concentrated force, the n -th order moment is expressed as $M^{(n)} = F[-(x_L - x)]^n$, while for concentrated moment, it is expressed as $M^{(n)} = nM[-(x_L - x)]^{n-1}$. When n is odd, both expressions preserve the sign of the moment arm, and the resulting contributions remain consistent. However, when n is even, the two expressions no longer behave consistently: the force term involves an even power, which eliminates the sign of the moment arm, while the concentrated moment term involves an odd power, which preserves it. As a result, the contribution from different load types may acquire incompatible signs, leading to an incorrect overall result.

An alternative approach was considered, in which the moment arm is always taken as positive and the sign of the moment is determined solely by its base form, defined as positive when it produces tension at the bottom fiber. However, this approach also leads to inconsistency, as the resulting rotation no longer follows the classical sign convention for slope (rotation), where a positive value corresponds to a tangent line with upward inclination to the right.

For these reasons, consistent reference must be maintained throughout the formulation. For clarity and uniformity within the scope of this work, the sign conventions used are as follows:

- all points of interest are to be set rightwards of the point of origin;
- in the computation of higher-order moments, forces (including internal forces such as shear and concentrated moments) that will cause tension at the bottom fiber and compression at the top have a positive sign, and vice versa;
- for angle of rotation, counterclockwise rotation is positive, and vice versa;
- for deflection, upward is positive, and vice versa.

Statically Indeterminate Beams

For statically indeterminate systems, the proposed method is particularly useful. The elastic curve equations can be used to generate compatibility conditions, which supplement the standard equilibrium equations. This enables the resolution of unknown reactions using purely analytical expressions derived from beam mechanics (see Example 4).

RESULTS AND DISCUSSION

Worked examples are presented to demonstrate the application of the proposed method. The results obtained using the higher-order moment approach are compared with classical solutions to confirm the validity and accuracy of the derived equations. The method consistently produces results identical to those obtained using traditional techniques while simplifying the computation process.

Example 1: Cantilever Beam with Discontinuous Uniform Load

Problem Statement:

Determine the rotation and deflection at point B , located 5.2 m from the fixed end of a cantilever beam subjected to a discontinuous uniform load, as shown in Fig. 6. Assume constant flexural rigidity EI .

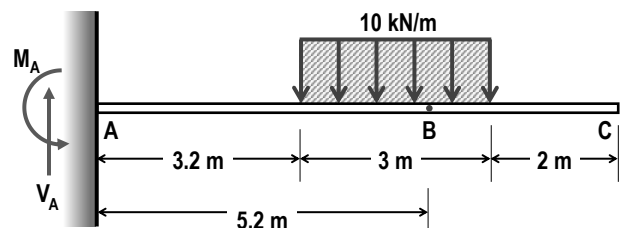


Figure 6: Cantilever Beam under a Discontinuous Uniformly Distributed Load.

Solution:

Step 1: Identify support conditions and reactions

Since this is a cantilever beam fixed at point A , the boundary conditions are as follows:

$$\begin{aligned}\delta_A &= 0 \\ \theta_A &= 0\end{aligned}$$

Cantilever beams are statically determinate. The support reactions were calculated using equations of static equilibrium and were found to be:

$$\begin{aligned}R_A &= 30 \text{ kN} \\ M_A &= 141 \text{ kN} \cdot \text{m}\end{aligned}$$

Step 2: Defining segment for analysis

Set point A as the origin, and point B as the point of interest, see Fig. 7. The step includes determining all applied loads and reactions relevant for the calculation, such as the value of shear and moment at the point of origin, A , which was already solved in step 1.

Note that all higher-order moments in the calculation are taken about the point of interest, B . This is the reason why shear and moment at the point of interest, B , need not be solved, because the moment arm of these forces with respect to the point of interest, B , is zero, and therefore will have no contribution to the summation of higher-order moments of force in the computation of deflection and rotation.

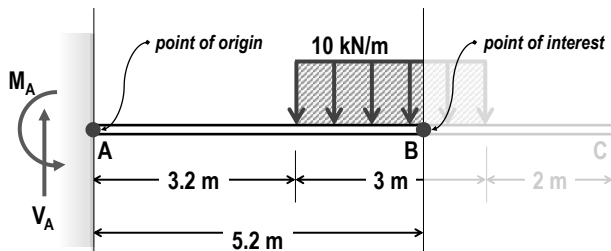


Figure 7: Figure showing the segment to be analyzed for Example 1.

Step 3: Compute second moment of force (for rotation)

Theorem I is used to solve for the rotation at B.

Using Theorem I, elastic equation for rotation of the beam is written as:

$$\theta_B - \theta_A = \frac{1}{2EI} \sum M_B^{(2)}$$

Substituting known values such as $\theta_A = 0$, and computing second-order moment contributions of each force within segment AB, about point B:

$$\theta_B - 0 = \frac{1}{2EI} \left[(30)(5.2)^2 - 2(141)(5.2) - \int_{3.2}^{5.2} 10(5.2 - x)^2 dx \right]$$

Simplifying,

$$\theta_B = \frac{-340.933}{EI} \text{ kN} - \text{m}^2$$

(The negative sign indicates clockwise rotation.)

Step 4: Compute third moment of force (for deflection)

Theorem II was used to solve for the deflection at B. the elastic curve for deflection for the beam is written as:

$$\delta_B - \delta_A = \theta_A L + \frac{1}{6EI} \sum M_B^{(3)}$$

Substituting known values and computing third-order moment contributions of each force within segment AB, about point B:

$$\delta_B - 0 = 0 + \frac{1}{6EI} \left[(30)(5.2)^3 - 3(141)(5.2)^2 - \int_{3.2}^{5.2} 10(5.2 - x)^3 dx \right]$$

Simplifying,

$$\delta_B = \frac{-1209.947}{EI} \text{ kN} - \text{m}^3$$

(The negative sign indicates a downward deflection.)

The results were verified using classical methods, specifically the moment-area and the double-integration methods, and were found to be consistent with the results presented above.

Example 2: Cantilever Beam with Support at the Right End

To demonstrate that the unknown deflection and rotation may not always be at the “point of interest” and that it can sometimes be at the “point of origin”, consider horizontally flipping the beam in Example 1.

Problem Statement:

Determine the rotation and deflection at point B, located 5.2 m from the fixed end of a cantilever beam subjected to a discontinuous uniform load, as shown in Fig. 8. Assume constant flexural rigidity EI .

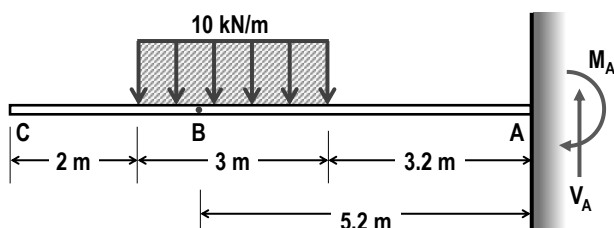


Figure 8: Cantilever Beam with Support at the Right End.

Solution:

Step 1: Identify support conditions and reactions

Since this is a cantilever beam fixed at point A, boundary conditions will include the following:

$$\delta_A = 0$$

$$\theta_A = 0$$

Cantilever beams are also statically determinate. Reactions at the support can be calculated using equations of static equilibrium and were found to be:

$$R_A = 30 \text{ kN}$$

$$M_A = 141 \text{ kN} - \text{m}$$

Step 2: Defining segment for analysis

Since the task is to solve for the deflection and rotation at point B, one might designate it as the point of interest, making point C the point of origin – since point of origin should always be to the left of point of interest. There is a major issue if segment CB is chosen for the analysis. No boundary conditions can be found in that segment, so both end points have unknown deflection and rotation.

In determining the segment to be used in the analysis, one should consider the following points as the end-points of the segment: (1) the point at which the deflection and/or the rotation is tasked to be solved; (2) the point at which the known value of deflection and/or rotation is located (typically the point where the boundary conditions can be found). Those points are point B and A, respectively. Therefore, the segment to be analyzed for this problem is segment BA. Since point of origin is always at the left-end and point of interest is always at the right, point B is designated as the point of origin and point A, on the other hand, is designated as point of interest, as shown in Fig. 9.

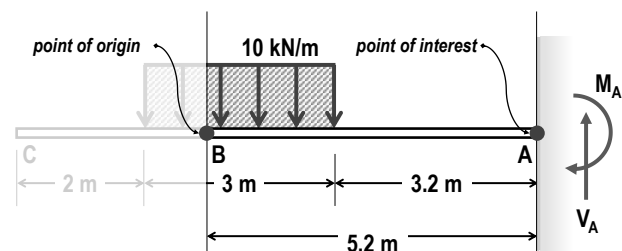


Figure 9: Figure showing the segment to be analyzed for Example 2.

Note that the shear and moment at the point of origin are always relevant in the calculation of deflection and rotation. In contrast, the shear and moment at the point of interest are not relevant in the calculation as explained in the previous problem.

To solve for the shear and moment at the point of origin, B, the beam was cut at that point and segment CB was isolated as shown in Fig. 10. This isolation was performed only to calculate the internal shear force and bending moment at point B; segment CB was not used for the deflection and rotation analysis. The shear and moment at point B were calculated and were found to be 10 kN and 5 kN – m, respectively.

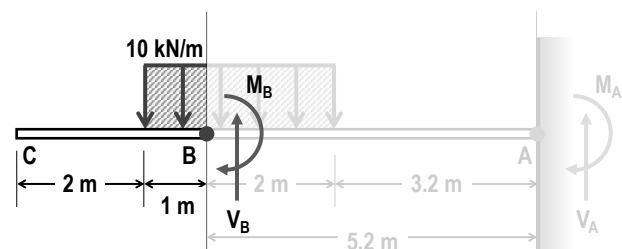


Figure 10: Figure showing segment CB of the beam in Example 2

$$\sum M_A = 0 : \quad M_A = \left(10 \frac{\text{kN}}{\text{m}} \right) (1 \text{ m}) \left(\frac{1 \text{ m}}{2} \right) = 5 \text{ kN} - \text{m}$$

The shear and moment at B are carried over to segment BA. See Fig. 11.

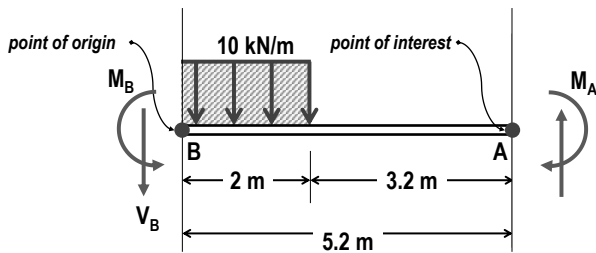


Figure 11: Figure showing all relevant forces for the analysis of the beam in Example 2.

Step 3: Compute second moment of force (for rotation)

Theorem I was applied to solve for the rotation at B.

Note that point B is the point of origin and point A is the point of interest. With this, elastic curve for rotation for the beam is written as:

$$\theta_A - \theta_B = \frac{1}{2EI} \sum M_A^{(2)}$$

Substituting known values such as $\theta_A = 0$, and computing second-order moment contributions of each force within segment BA, about point B:

$$0 - \theta_B = \frac{1}{2EI} \left[-(10)(5.2)^2 - 2(5)(5.2) - \int_0^{5.2} 10(5.2 - x)^2 dx \right]$$

Simplifying,

$$\theta_B = \frac{+340.933}{EI} \text{ kN} - \text{m}^2$$

(The positive sign indicates counterclockwise rotation.)

Step 4: Compute third moment of force (for deflection)

Theorem II was applied to solve for the deflection at B. The elastic curve for deflection is written as:

$$\delta_A - \delta_B = \theta_B L + \frac{1}{6EI} \sum M_A^{(3)}$$

Substituting known values and computing third-order moment contributions of each force within segment BA, about point B:

$$0 - \delta_B = \left(\frac{340.933}{EI} \right) (5.2) + \frac{1}{6EI} \left[-(10)(5.2)^3 - 3(5)(5.2)^2 - \int_0^{5.2} 10(5.2 - x)^3 dx \right]$$

Simplifying,

$$\delta_B = \frac{-1209.947}{EI} \text{ kN} - \text{m}^3$$

(The negative sign indicates a downward deflection.)

The results were verified using classical methods, specifically the moment-area method and double-integration method, and were found to be consistent with the results presented above.

Example 3: Simple Beam

Problem Statement:

Determine the deflection and rotation at the midspan of the simply supported beam shown in Fig. 12. Assume EI is constant.

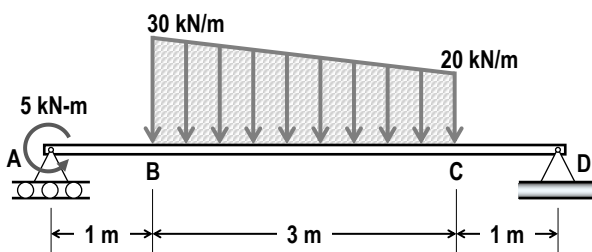


Figure 12: Simple beam under a discontinuous varying load and a concentrated moment load.

Solution:

Step 1: Identify support conditions and reactions

Since this is a simply supported beam, boundary conditions will include zero deflections at the supports:

$$\begin{aligned} \delta_A &= 0 \\ \delta_D &= 0 \end{aligned}$$

Reactions at the support were calculated using equations of static equilibrium and were found to be:

$$\begin{aligned} R_A &= 40 \text{ kN} \\ R_D &= 35 \text{ kN} \end{aligned}$$

Step 2: Defining segment for analysis

For a simply supported beam, the boundary conditions are specified at two distinct points, with both indicating zero deflection at the supports. This condition is not ideal, as the elastic curve equations require that at least one angle of rotation be known to fully solve for the deflection and rotation. For this reason, the rotation at the point of origin, A, was solved first. This is done by applying the elastic curve equation for deflection, and using the boundary conditions at the supports, point A and point D, as the point of origin and point of interest, respectively.

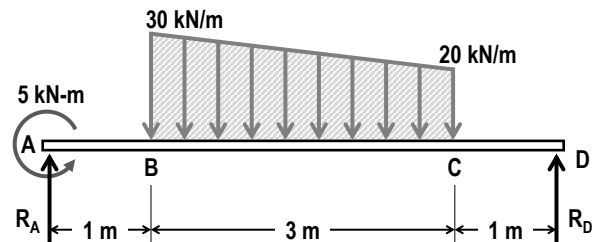


Figure 13: Figure showing all relevant forces for the analysis of the beam in Example 3.

From Theorem II, the elastic curve equation for the beam deflection is written as:

$$\delta_D - \delta_A = \theta_A L + \frac{1}{6EI} \sum M_D^{(3)}$$

Substituting the known values from the beam shown in Fig. 13,

$$0 - 0 = \theta_A (5) + \frac{1}{6EI} \left[(40)(5)^3 - 3(5)(5)^2 - \int_1^4 \left(30 - \left(\frac{30-20}{4-1} \right) (x-1) \right) (5-x)^3 dx \right]$$

Simplifying,

$$\theta_A = -\frac{96.017}{EI} \text{ kN} - \text{m}^2$$

The value of θ_A will now be used to solve what is asked in the problem.

Step 3: Compute second moment of force (for rotation)

Theorem I is applied to solve for the angle of rotation at midspan.

Note that point A is the point of origin, and the midspan, denoted as E, is the point of interest as shown in Fig. 14.

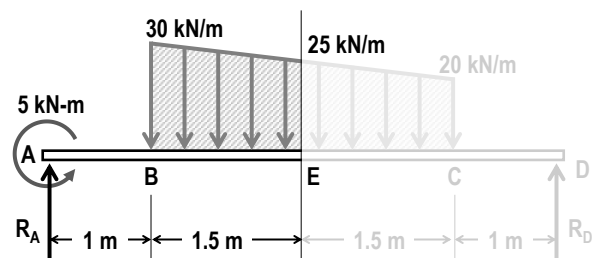


Figure 14: Figure showing all relevant forces in segment AE of the beam in Example 3.

From Theorem I, elastic equation for rotation is rewritten as:

$$\theta_E - \theta_A = \frac{1}{2EI} \sum M_E^{(2)}$$

Substituting the value of θ_A and computing second-order moment contributions of each force within segment AE, about point E:

$$\begin{aligned} \theta_E - \left[-\frac{96.017}{EI} \right] &= \frac{1}{2EI} \left[(40)(2.5)^2 - 2(5)(2.5) \right. \\ &\quad \left. - \int_1^{2.5} \left(30 - \left(\frac{30-20}{4-1} \right) (x-1) \right) (2.5-x)^2 dx \right] \end{aligned}$$

Simplifying,

$$\theta_E = \frac{+0.311}{EI} \text{ kN} - \text{m}^2$$

(The positive sign indicates counterclockwise rotation.)

Step 4: Compute third moment of force (for deflection)

Theorem II was used to solve for the deflection at E. The elastic equation for deflection for the beam is written as:

$$\delta_E - \delta_A = \theta_A L + \frac{1}{6EI} \sum M_E^{(3)}$$

Substituting known values and computing third-order moment contributions of each force within segment AE, about point E:

$$\delta_E - 0 = \left[-\frac{96.017}{EI} \right] (2.5) + \frac{1}{6EI} \left[(40)(2.5)^3 - 3(5)(2.5^2) - \int_1^{2.5} \left(30 - \left(\frac{30-20}{4-1} \right) (x-1) \right) (2.5-x)^3 dx \right]$$

Simplifying,

$$\delta_E = \frac{-157.617}{EI} \text{ kN} - \text{m}^3$$

(The negative sign indicates a downward deflection.)

The results were verified using classical methods, specifically the moment-area and virtual work methods, and were found to be consistent with the results presented above.

Example 4: Beam with fixed support at both ends

Problem Statement:

Determine the deflection and rotation at point B of a beam that is fixed at both ends, as shown in Fig. 15. Also determine the maximum deflection. Assume EI is constant.

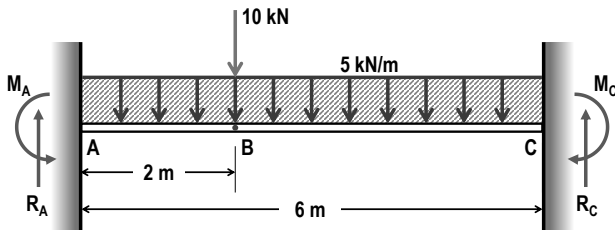


Figure 15: Fixed-ended beam under a uniformly distributed load and a concentrated load.

Solution:

Step 1: Identify support conditions and reactions

For a beam that is fixed at both ends, the deflection and angle of rotation at both supports are all equal to zero. Thus:

$$\theta_A = 0; \delta_A = 0; \theta_C = 0; \delta_C = 0$$

The beam is statically indeterminate; therefore, equations of static equilibrium alone cannot solve for the reactions. Theorems I and II provide additional equations for the unknown reactions.

From Theorem I (rotation), the elastic curve function is written as:

$$\theta_C - \theta_A = \frac{1}{2EI} \sum M_C^{(2)}$$

$$0 - 0 = \frac{1}{2EI} \left[R_A(6^2) - 2M_A(6) - 10(6-2)^2 - \int_0^6 5(6-x)^2 dx \right]$$

Simplifying,

$$36R_A - 12M_A = 520 \rightarrow eq. A$$

From Theorem II (deflection), the elastic curve function is written as:

$$\delta_C - \delta_A = \theta_A L + \frac{1}{6EI} \sum M_C^{(3)}$$

$$0 - 0 = (0)L + \frac{1}{6EI} \left[R_A(6^3) - 3M_A(6^2) - 10(6-2)^3 - \int_0^6 5(6-x)^3 dx \right]$$

Simplifying,

$$216R_A - 108M_A = 2260 \rightarrow eq. B$$

Solving eq. A and eq. B simultaneously using system of linear equations yields

$$R_A = \frac{605}{27} \text{ kN}; M_A = \frac{215}{9} \text{ kN} - \text{m}$$

Step 2: Solving Rotation and Deflection at Point B

Point A is taken as the origin and B as the point of interest; segment AB is analyzed (Fig. 16).

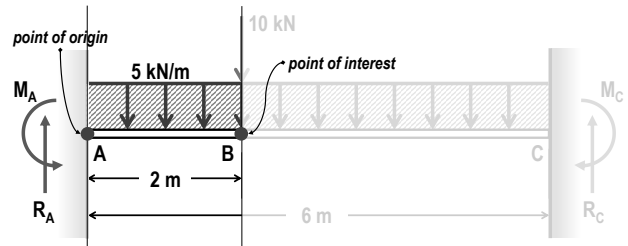


Figure 16: Figure showing all relevant forces in segment AB of the beam in Example 4.

From Theorem I, the elastic curve for rotation is written as:

$$\theta_B - \theta_A = \frac{1}{2EI} \sum M_B^{(2)}$$

$$\theta_B - 0 = \frac{1}{2EI} \left[\left(\frac{605}{27} \right) (2^2) - 2 \left(\frac{215}{9} \right) (2) - \int_0^2 5(2-x)^2 dx \right]$$

Simplifying,

$$\theta_B = -\frac{9.630}{EI} \text{ kN} - \text{m}^2$$

(The negative sign indicates clockwise rotation.)

From Theorem II, elastic equation for deflection for the beam is written as:

$$\delta_B - \delta_A = \theta_A L + \frac{1}{6EI} \sum M_B^{(3)}$$

$$\delta_B - 0 = (0)(2) + \frac{1}{6EI} \left[\left(\frac{605}{27} \right) (2^3) - 3 \left(\frac{215}{9} \right) (2^2) - \int_0^2 5(2-x)^3 dx \right]$$

Simplifying,

$$\delta_B = -\frac{21.235}{EI} \text{ kN} - \text{m}^3$$

(The negative sign indicates a downward deflection.)

Step 3: Compute the maximum deflection

Theorems I and II can be used jointly to determine the maximum deflection of the beam. The procedure begins by applying Theorem I to identify the location at which the maximum deflection occurs. This location corresponds to the point where the rotation is zero. Let this point of interest be located at an unknown position, denoted by x_L . Assuming that the maximum deflection occurs at a point to the right of point B (i.e., $x_L > 2$), the elastic curve equation for rotation may be written as:

$$\theta_{x_L} - 0 = \frac{1}{2EI} \left[\left(\frac{605}{27} \right) (x_L)^2 - 2 \left(\frac{215}{9} \right) (x_L) - 10(x_L-2)^2 - \int_0^{x_L} 5(x_L-x)^2 dx \right]$$

Set $\theta_{x_L} = 0$ and simplifying yields

$$0 = \frac{5}{3} (x_L)^3 - \frac{335}{27} (x_L)^2 + \frac{70}{9} (x_L) + 40$$

The resulting equation is cubic, indicating that up to three possible values of x_L may exist. Solving the equation gives

$$x_{L1} = 6 \text{ m.}$$

$$x_{L2} = -1.404 \text{ m.}$$

$$x_{L3} = 2.849 \text{ m.}$$

Among these values, x_{L2} , which was equal to -1.404 m , was disregarded because it lay outside the beam span. Since the beam has a total length of 6 m and the origin is located at $x = 0$, the valid range of x_L is $0 \leq x_L \leq 6$. The solution $x_{L1} = 6 \text{ m}$ corresponds to point C, where the rotation is zero by the boundary conditions established in Step 1, and therefore does not represent the location of maximum deflection within the span. The remaining solution, $x_{L3} = 2.849 \text{ m}$, was verified against the initial assumption that the point of maximum deflection lay to the right of point B (i.e., $x_L > 2$). Since this condition is satisfied, the solution was accepted.

Thus, the location at which the maximum deflection occurs was:

$$x_L = 2.849 \text{ m.}$$

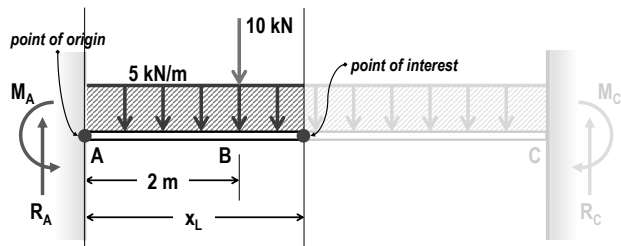


Figure 17: Figure showing the location of maximum deflection in Example 4.

From Theorem II, the elastic curve equation for deflection is expressed as

$$\delta_{x_L} - 0 = (0)(x_L) + \frac{1}{6EI} \left[\left(\frac{605}{27} \right) (x_L)^3 - 3 \left(\frac{215}{9} \right) (x_L)^2 - 10(x_L - 2)^3 - \int_0^{x_L} 5(x_L - x)^3 dx \right]$$

Substituting the value of $x_L = 2.849$ m and simplifying,

$$\delta_{max} = \delta_{x_L} = -\frac{25.335}{EI} \text{ kN} - \text{m}^3$$

(The negative sign indicates a downward deflection.)

The results were verified using classical methods, specifically moment-area and conjugate-beam methods, and were found to be consistent with the results presented above.

Example 5: Beam with moment release

Problem Statement:

Determine the deflection and rotation at point C of the beam shown on Fig. 18. Assume EI is constant.

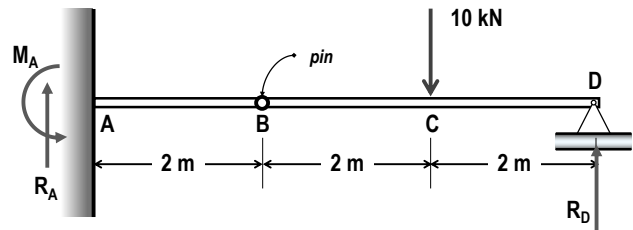


Figure 18: Beam with internal pin support.

Solution:

Step 1: Identify support conditions and reactions

The deflection and angle of rotation at the fixed support A are both equal to zero and the deflection at the external hinged support D is also equal to zero. Thus:

$$\theta_A = 0; \quad \delta_A = 0; \quad \delta_D = 0$$

This type of beam is statically determinate; therefore, the reactions at the supports can be determined using the equations of static equilibrium alone. The analysis strategy involves discretizing the beam into two segments, AB and BD, as shown in Figs. 19(A) and 19(B).

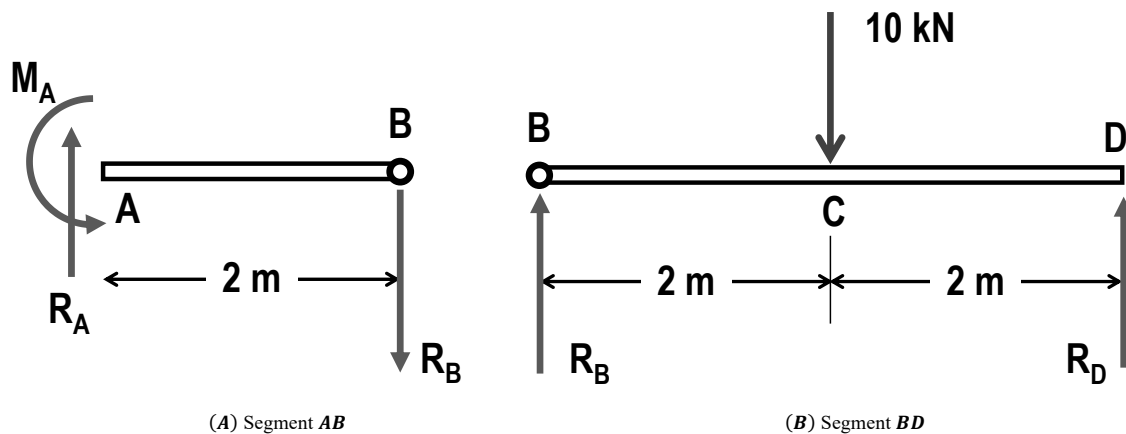


Figure 19: Free-body diagram of discretized members.

Using FBD for segment BD, shown in Fig. 19(B), the magnitude of the reactions R_B and R_D are solved. Results are

$$R_B = 5 \text{ kN}; \quad R_D = 5 \text{ kN}$$

The FBD for segment AB, shown in Fig. 19(A), is then used to solve the magnitude of the reactions R_A and M_A . Results are

$$R_A = 5 \text{ kN}; \quad M_A = 10 \text{ kN} - \text{m}$$

Step 2: Solve for the deflection and rotation at the internal pin support

Note that the slope (rotation) of the elastic curve at the internal pin B is discontinuous. Consequently, the rotation at B as obtained from segment AB differs from that obtained from segment BD. Because of this discontinuity, a single elastic curve equation cannot describe the entire beam. Instead, separate elastic curve equations must be formulated and solved for segments AB and BD.

Theorem I is applied to determine the angle of rotation at point B for segment AB. The elastic curve equation for rotation is written as

$$\theta_B - \theta_A = \frac{1}{2EI} \sum M_B^{(2)}$$

Substituting the known values from the beam shown in Fig. 19(A),

$$\theta_B - 0 = \frac{1}{2EI} [5(2)^2 2(10)(2)]$$

Simplifying,

$$\theta_B = -\frac{10}{EI} \text{ kN} - \text{m}^2$$

(The negative sign indicates clockwise rotation.)

Theorem II is then applied to solve for the deflection at point B in segment AB. The elastic equation for the said deflection is written as

$$\delta_B - \delta_A = \theta_A L + \frac{1}{6EI} \sum M_B^{(3)}$$

Substituting the known values from the beam shown in Fig. 19(A),

$$\delta_B - 0 = (0)(2) + \frac{1}{6EI} [5(2)^3 - 3(10)(2)^2]$$

Simplifying,

$$\delta_B = -\frac{40}{3EI} \text{ kN} - \text{m}^3$$

(The negative sign indicates downward deflection.)

Theorem II is also applied to solve for the rotation at point B in segment BD. The elastic equation from theorem II is written as

$$\delta_D - \delta_B = \theta_B L + \frac{1}{6EI} \sum M_D^{(3)}$$

Substituting the known values from the beam shown in Fig. 19(B),

$$0 - \left(-\frac{40}{3EI} \right) = (0)(2) + \frac{1}{6EI} [5(4)^3 - (10)(2)^3]$$

Simplifying,

$$\theta_B = -\frac{20}{3EI} \text{ kN} - m^2$$

(The negative sign indicates clockwise rotation.)

Step 3: Solve for the deflection and rotation at C

In solving for the deflection and rotation of the beam at point C, FBD of segment BD is used (see Fig. 19(B)).

Theorem I is applied to solve for the angle of rotation at point C. The elastic equation for rotation for the beam segment is written as

$$\theta_C - \theta_B = \frac{1}{2EI} \sum M_C^{(2)}$$

Substituting the known values from the beam shown in Fig. 19(B),

$$\theta_C - \left(-\frac{20}{3EI}\right) = \frac{1}{2EI} [5(2)^2]$$

Simplifying,

$$\theta_C = +\frac{10}{3EI} \text{ kN} - m^2$$

(The positive sign indicates counterclockwise rotation.)

Theorem II is applied to solve for the deflection at point C in segment BD. The elastic equation for the said deflection is written as

$$\delta_C - \delta_B = \theta_B L + \frac{1}{6EI} \sum M_C^{(3)}$$

Substituting the known values from the beam shown in Fig. 19(B),

$$\delta_C - \left(-\frac{40}{3EI}\right) = \left(-\frac{20}{3EI}\right)(2) + \frac{1}{6EI} [5(2)^3]$$

Simplifying,

$$\delta_C = -\frac{20}{EI} \text{ kN} - m^3$$

(The negative sign indicates downward deflection.)

The results were verified using classical methods, specifically the double-integration and moment-area methods, and were found to be consistent with the results presented above.

Example 6: Beam with shear release

Problem Statement:

Determine the deflection at point B of a beam shown in Fig. 20 if vertical translation is allowed but the beam is restrained for rotation at both supports. The beam is fully restrained at A. Assume EI is constant.

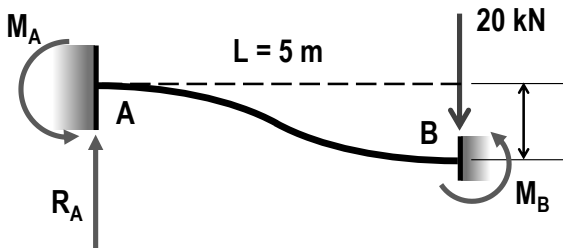


Figure 20: Beam with guided support.

Solution:

Step 1: Identify support conditions and reactions

The deflection and angle of rotation at the fixed support A are both equal to zero. At support B, only the rotation is equal to zero because the beam is allowed to deflect at that point. Thus:

$$\theta_A = 0; \delta_A = 0; \theta_B = 0$$

There is no vertical reaction at point B since shear is released at that point. Therefore, the vertical reaction at point A can be determined using the equations of static equilibrium alone. The result is

$$R_A = 10 \text{ kN}$$

Theorem I is applied to solve for the moment at point A. The elastic equation for rotation of the beam is written as

$$\theta_B - \theta_A = \frac{1}{2EI} \sum M_B^{(2)}$$

Substituting the known values from the beam shown in Fig. 20,

$$0 - 0 = \frac{1}{2EI} [10(5)^2 - 2(M_A)(5)]$$

Solving for M_A ,

$$M_A = 25 \text{ kN} - m$$

Step 2: Solve for the deflection at B

Theorem II is applied to solve for the deflection at point B. The elastic equation for the said deflection is written as

$$\delta_B - \delta_A = \theta_A L + \frac{1}{6EI} \sum M_B^{(3)}$$

Substituting the known values from the beam shown in Fig. 20,

$$\delta_B - 0 = (0)(5) + \frac{1}{6EI} [10(5)^3 - 3(25)(5)^2]$$

Simplifying,

$$\delta_B = -\frac{104.167}{EI} \text{ kN} - m^3$$

(The negative sign indicates downward deflection.)

The results were verified using classical methods, specifically the moment-area method, and were found to be consistent with the results presented above.

CONCLUSION

This paper presented a novel analytical framework for evaluating beam rotation and deflection based on higher-order moments of force. Unlike traditional approaches that rely on segment-specific moment expressions, auxiliary diagrams, and integration steps, the proposed method derives closed-form elastic curve equations using the second- and third-order moments of internal and external forces about a reference point. The method simplifies the analysis by eliminating the need for shear and moment diagrams or indefinite integrations, enabling direct computation of slope and deflection from force magnitudes and their locations.

The applicability of the proposed framework extends to a wide range of beam configurations through Theorems I and II, including prismatic and piecewise-prismatic members subjected to concentrated forces, distributed loads, and applied moments. Boundary conditions involving moment release and shear release are explicitly addressed, as demonstrated in Example 5, which examines a beam with an internal pin (moment-release condition), and Example 6, which considers a beam with shear-release boundary conditions. In both cases, the proposed formulation correctly enforces the corresponding boundary constraints and yields results that are consistent with classical beam theory.

To further demonstrate the robustness of the method, a statically indeterminate beam with fixed-fixed support conditions was analyzed in Example 4. In this example, the proposed framework was used to determine support reactions, rotations, deflections, and the location and magnitude of maximum deflection using compatibility conditions derived from the theorems, without requiring auxiliary diagrams. This confirms that the method is suitable for beams with fully restrained boundary conditions and can be applied effectively to indeterminate systems.

Although higher-order moments of force lack a physical interpretation, their mathematical formulation is shown to be rigorous and consistent with classical beam theory, providing accurate results with reduced computational effort. The derived elastic curve equations may also be used to formulate compatibility conditions in statically indeterminate systems, demonstrating the method's versatility beyond determinate structures. Furthermore, the formal definition and application of second- and third-order moments of force establish a foundation for future research, potentially supporting advancements in structural modeling, computational tools, or instructional techniques.

Despite its generality, the proposed method is subject to several limitations. The formulation is grounded in Euler–Bernoulli beam theory and therefore neglects shear deformation and rotary inertia; as with other classical beam deflection methods based on the same theory—such as the double-integration, moment-area, and conjugate-beam methods—its accuracy may decrease for deep beams or short spans where shear effects become significant. In addition, the method assumes linear elastic behavior and piecewise-constant flexural rigidity and is not intended for geometrically nonlinear problems or beams undergoing large deflections. These limitations define the current scope of applicability and indicate directions for future extensions.

Overall, the proposed framework offers a unified, efficient, and generalizable alternative to conventional deflection analysis methods, with practical value for both structural engineering practice and education.

ACKNOWLEDGMENT

The authors gratefully acknowledge Eastern Visayas State University, specifically the School of Engineering and the Civil Engineering Department, for the administrative and academic support that made this study possible. The authors also wish to thank their families for their continuous moral support and understanding during the preparation of this manuscript.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

CONTRIBUTIONS OF INDIVIDUAL AUTHORS

Lee Justin P. Rondina: Conceived the study, developed the analytical framework, performed the main mathematical derivations, and prepared the initial manuscript draft.

Osmer Gabriel A. Deypalan: Contributed to the study design, assisted in the development and verification of theorems and proofs, and participated in drafting and revising the manuscript.

Hazel O. Cadiente: Generated and analyzed worked examples to validate the proposed method, contributed to the interpretation of results, and provided critical feedback on the manuscript.

Julimar S. Labor: Assisted in verifying the accuracy of derivations and equations, ensured consistency of figures and tables, and provided substantial revisions for clarity and structure.

Mark Kevin T. Cidro: Supervised the overall study, provided guidance on methodology and presentation, critically reviewed the manuscript, and prepared the final version for submission.

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